

Radiation Damping and Quantum Excitation for Longitudinal Charged Particle Dynamics in the Thermal Wave Model

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Abstract

On the basis of the recently proposed *Thermal Wave Model (TWM)* for *particle beams*, we give a description of the longitudinal charge particle dynamics in circular accelerating machines by taking into account both radiation damping and quantum excitation (stochastic effect), in presence of a RF potential well. The longitudinal dynamics is governed by a 1-D Schrödinger-like equation for a complex wave function whose squared modulus gives the longitudinal bunch density profile. In this framework, the appropriate *r.m.s. emittance* scaling law, due to the damping effect, is naturally recovered, and the asymptotic equilibrium condition for the bunch length, due to the competition between quantum excitation (QE) and radiation damping (RD), is found. This result opens the possibility to apply the TWM, already tested for protons, to electrons, for which QE and RD are very important.

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In the study of charged particle beam dynamics for accelerators and plasma physics, a number of nonlinear and collective effects are relevant [1]. Due to the electromagnetic interactions between the particles and their image charges induced in the surroundings, these nonlinear effects also acquire collective nature [1]. This property is enhanced for very intense beams, which are employed in very high luminosity colliders. In addition, radiation damping and quantum electromagnetic fluctuations (quantum excitation) are generally present in the beam longitudinal dynamics, and, in particular, for electron circular accelerating machines are not negligible [2].

Recently, a *Thermal Wave Model* (TWM) for charged particle beam dynamics has been formulated [3] and successfully applied to a number of linear and nonlinear problems in beam physics [4]-[9]. In this approach, the beam transverse (longitudinal) dynamics is formulated in terms of a complex function, the so called *beam wave function* (BWF), whose squared modulus is proportional to the bunch density. This wave function satisfies a Schrödinger-like equation in which Planck's constant is substituted with the transverse (longitudinal) bunch emittance [3],[8].

In particular, this model is capable of reproducing the main results of the conventional theory about transverse beam optics and dynamics (in linear and nonlinear devices) [3], and it represents a new approach for estimating the luminosity in particle accelerators [5], [6], as well as to study the self-consistent beam-plasma interaction [4]. As far as the longitudinal bunch dynamics is concerned, in this scheme one can describe, in a simple way, the synchrotron motion when both self-interaction and the radio frequency (RF) potential well are taken into account. In particular, the right conditions for the coherent instability in circular machines have been recovered [7],[8] and soliton-like solutions for the beam density have been discovered [7]-[9].

In this letter we improve the *thermal wave model for longitudinal bunch dynamics* given in [7]-[9]. By starting from the conventional longitudinal single-particle dynamics in circular accelerators, the problem under study is formulated in terms of an appropriate wave model which describes the evolution of the beam, when the RF potential well is taken into account together with radiation damping and quantum excitation. We show that the longitudinal beam dynamics is still correctly governed by a Schrödinger-like equation for the BWF. The envelope description is straightforwardly obtained from the wave solution and, correspondingly, the results are compared with those that are given in the conventional theory. In particular, an asymptotic time-limit for the bunch length and the r.m.s. emittance scaling law are obtained.

Let us consider the motion of a single particle within a stationary bunch, travelling with velocity βc ($\beta \approx 1$) in a circular accelerating machine of radius $R_0 = cT_0/2\pi$ (T_0 being the revolution period). It is well-known that if both radiation damping and quantum excitation are taken into account, defining x as the longitudinal displacement of the particle with respect to the synchronous one, and $s \equiv ct \geq cT_0$ (t being the time), the particle dynamics is governed by the following set of equations [10]:

$$\frac{dx}{ds} = \eta \mathcal{P} \quad , \quad (1)$$

$$\frac{d\mathcal{P}}{ds} = -\frac{q\Delta V}{cT_0 E_0} - \gamma \mathcal{P} - \frac{1}{E_0} \frac{dR}{ds} \quad , \quad (2)$$

where $\mathcal{P} \equiv \frac{\Delta E}{E_0}$ denotes the relative longitudinal energy spread of the single particle, computed with respect to the synchronous one ($\Delta E = 0$), and $U(x, s) \equiv (1/cT_0E_0) \int_0^x q \Delta V(y) dy$ is the effective potential energy that the particle sees after a turn in the ring (ΔV being the corresponding total voltage variation seen by the particle). Moreover, in Eqs. (1)-(2) E_0 and q stand for the synchronous particle energy and charge respectively, and $\eta \equiv (1 - \beta^2) - \alpha_c$ is the phase slip factor (α_c is the momentum compaction)[1]. Finally, in (2), γ represents the damping coefficient [10] and dR/ds accounts for the quantum excitation effect (noise), $R(s)$ being the difference between the energy effectively radiated by the particle during the time interval s/c and the average of this energy. Since $R(s)$ is a stochastic quantity, namely its average value vanishes whilst its r.m.s. is not zero, we cannot treat the force term dR/ds as the other ones of r.h.s. in (2), which on the contrary have deterministic nature. Hereafter, without loss of generality, we restrict our analysis to electron machines ($q = e$, $|\eta| \approx \alpha_c$) for which the e.m. emission is a relevant phenomenon.

The r.m.s. of $R^2(s)$, $\overline{R^2}$, obeys to the following equation [2]

$$\frac{d\overline{R^2}}{ds} + \gamma \overline{R^2} = Q_p \quad , \quad (3)$$

with

$$Q_p \equiv \frac{\rho_0}{R_0} \frac{N_\gamma}{c} \langle u^2 \rangle = \frac{55}{24\sqrt{3}} \frac{\hbar c r_e E_0 \gamma_0^6}{R_0 \rho_0^2} \quad , \quad (4)$$

where ρ_0 is the magnetic bending radius, $\langle u^2 \rangle^{1/2}$ is the quantum fluctuations associated to the noise, N_γ is the mean rate of photon emission, r_e is the classical electron radius, \hbar is the Planck's constant, and $\gamma_0 \equiv (1 - \beta^2)^{-1/2}$ is the relativistic factor.

It is easy to see that (1) and (2) are the usual equations for the longitudinal motion [10] under the substitution $s = ct$ and $\mathcal{P} = \Delta E/E_0$.

It is worth to point out that the distinction between deterministic and stochastic force terms is a necessary requirement to properly construct the wave equation ruling the collective behaviour of the beam. This difference can be understood observing how the emittance, which also describes a stochastic effect but related to the temperature of the system, is involved in a wave equation (Schrödinger-like equation)[3]-[9]. In fact, despite of deterministic terms, the emittance is the only quantity which plays the role analogous to a diffraction parameter. In the framework of TWM this parameter is involved in the quantization rules.

To understand how to include quantum excitation in the TWM description, we first consider the simplified situation in which quantum excitation is negligible. Then, we generalize our results by taking into account also this effect. Under this hypothesis, by considering a linearized RF-voltage only ($U(x, s) = U_{RF}(x) \approx (K/2\eta) x^2$, where K is the RF cavity strength, supposed for simplicity to be positive), it is easy to prove that the Lagrangian associated to (1) and (2) is given by [11], [12]

$$\mathcal{L}(x, x', s) = \frac{1}{2\eta} [x'^2 - K x^2] e^{\gamma s} \quad , \quad (5)$$

where $x' \equiv dx/ds$. Hence, computing the x -variable conjugate momentum by $p \equiv \partial \mathcal{L} / \partial x' = (x'/\eta) \exp(\gamma s)$, the corresponding hamiltonian results to be

$$\mathcal{H}(x, p, s) = \frac{\eta}{2} p^2 e^{-\gamma s} + \frac{1}{2} \frac{K}{\eta} x^2 e^{\gamma s} \quad . \quad (6)$$

In order to write a Schrödinger-like equation for the BWF, which describes the longitudinal dynamics of a short bunch ($\sigma \ll R_0$) in presence of the only radiation damping we have to follow the *Thermal Quantization Rules* (TQM), in complete analogy with our previous works [3]-[9] : $p \rightarrow \hat{p} \equiv -i\epsilon\partial/\partial x$, and $\mathcal{H} \rightarrow \hat{\mathcal{H}} \equiv i\epsilon\partial/\partial s$. Therefore, (6) gives (for $\eta \neq 0$)

$$i\eta\epsilon e^{-\gamma s} \frac{\partial}{\partial s} \Psi(x, s) = -\frac{\eta^2 \epsilon^2 e^{-2\gamma s}}{2} \frac{\partial^2}{\partial x^2} \Psi(x, s) + \frac{1}{2} K x^2 \Psi(x, s) , \quad (7)$$

where ϵ is a constant to be determined, which, according to Ref.s [3]-[9], accounts for thermal spreading of the bunch and plays the role fully similar to Planck's constant. The BWF $\Psi(x, s)$ satisfies the following normalization condition

$$\int_{-\infty}^{\infty} |\Psi(x, s)|^2 dx = 1 , \quad (8)$$

which, fixed for $s = 0$, holds for any s , due to hermiticity of the hamiltonian operator (6). Thus, according to the features of the TWM, if N is the total number of particles in the bunch, $N|\Psi(x, s)|^2$ represents the longitudinal bunch number density (number of particles per unity length).

Interestingly, a complete set of solutions of (7) is given in terms of Hermite-Gauss modes:

$$\begin{aligned} \Psi_m(x, s) = & \frac{1}{\{2\pi 2^{2m} (m!)^2 \sigma^2(s)\}^{1/4}} \exp\left\{-\frac{x^2}{4\sigma^2(s)}\right\} H_m\left(\frac{x}{\sqrt{2}\sigma(s)}\right) \\ & \times \exp\left\{i\frac{x^2}{2\eta\epsilon \exp(-\gamma s) \rho(s)} + i(1+2m)\phi(s)\right\} , \end{aligned} \quad (9)$$

where the H_m 's are the Hermite polynomials ($m = 0, 1, 2, \dots$), and the functions $\sigma(s)$, $\rho(s)$ and $\phi(s)$ satisfy the following set of differential equations

$$\frac{d^2\sigma(s)}{ds^2} + \gamma \frac{d\sigma(s)}{ds} + K\sigma(s) - \frac{\eta^2 \epsilon^2 e^{-2\gamma s}}{4\sigma^3(s)} = 0 . \quad (10)$$

$$\frac{1}{\rho} = \frac{1}{\sigma(s)} \frac{d\sigma(s)}{ds} , \quad (11)$$

$$\frac{d\phi}{ds} = -\frac{\eta\epsilon e^{-\gamma s}}{4\sigma^2(s)} . \quad (12)$$

In particular, taking (9) for $m = 0$ (fundamental mode), we note that $\sigma(s)$ results to be the corresponding bunch length, defined as

$$\sigma^2(s) = \int_{-\infty}^{+\infty} x^2 |\Psi_0(x, s)|^2 dx \equiv \langle x^2 \rangle . \quad (13)$$

In addition it is useful to define the momentum spread $\sigma_p(s)$

$$\sigma_p^2(s) = \epsilon^2 \int_{-\infty}^{+\infty} \left| \frac{\partial}{\partial x} \Psi_0(x, s) \right|^2 dx \equiv \langle \hat{p}^2 \rangle . \quad (14)$$

By using the definition of \hat{p} and of $\Psi_0(x, s)$, and the quantum formalism for the average of the operators, we can show that the following expression $A(s)$ is a constant of motion

$$A(s) \equiv 2 \left[\langle x^2 \rangle \langle \hat{p}^2 \rangle - \left(\frac{1}{2} \langle x\hat{p} + \hat{p}x \rangle \right)^2 \right]^{1/2} = \epsilon \quad , \quad (15)$$

and it coincides with the diffraction parameter ϵ of the TW. Observe that (15) is formally identical to the well-known Robertson-Schrödinger uncertainty relation [13],[14], and ϵ , which is straightforwardly obtained taking the minimum of this relation, is the natural extension to the quantum-like description of one of the Courant-Snyder invariants, well-known in particle accelerators [15] (Poincaré-Cartan invariants, in classical mechanics [16]).

From (15), we can also derive the scaling law for the following effective emittance $\bar{\epsilon}(s)$ (*quantum-like r.m.s. emittance*), defined in analogy to the classical definition of r.m.s. emittance given by Lapostolle [1],[17]

$$\bar{\epsilon}(s) \equiv \frac{2}{|\eta|} \left[\langle x^2 \rangle \langle x'^2 \rangle - \left(\frac{1}{2} \langle xx' + x'x \rangle \right)^2 \right]^{1/2} = \epsilon e^{-\gamma s} \quad . \quad (16)$$

The same scaling law can be extrapolated from Eq. (10) by comparing it with the corresponding expression for the undamped case [8], consequently the diffraction parameter of TW ϵ represents the initial value $\bar{\epsilon}(0)$. Furthermore, by virtue of (16), the envelope equation (10) could be rewritten substituting $\epsilon \exp(-\gamma s)$ with the effective emittance $\bar{\epsilon}(s)$. It is worth to point out that the presence of the friction-like term in (10) ($\gamma \neq 0$) is responsible for the s -dependence (time dependence) of r.m.s. emittance, whereas in the undamped case ($\gamma = 0$) $\bar{\epsilon}$ is a constant of motion and gives the accessible phase-space area associated with the system.

Remarkably, in order to obtain all the above results concerning with radiation damping we could start from the following Schrödinger-like equation

$$i\eta\bar{\epsilon}(s) \frac{\partial}{\partial s} \Psi(x, s) = - \frac{\eta^2 \bar{\epsilon}^2(s)}{2} \frac{\partial^2}{\partial x^2} \Psi(x, s) + \frac{1}{2} K x^2 \Psi(x, s) \quad , \quad (17)$$

formally obtained by introducing in (6) the following TQR: $p \rightarrow \hat{p} \equiv -i\bar{\epsilon}(s)\partial/\partial x$, and $\mathcal{H} \rightarrow \hat{\mathcal{H}} \equiv i\bar{\epsilon}(s)\partial/\partial s$, where $\bar{\epsilon}(s)$ is given by (16). For the present case we note that:

a) $\bar{\epsilon}(s)$ satisfies the following differential equation

$$\frac{d\bar{\epsilon}}{ds} + \gamma\bar{\epsilon} = 0 \quad (18)$$

with the initial condition $\bar{\epsilon}(0) = \epsilon$;

b) for this new TQR the Robertson-Schrödinger-like relation (15) does not give a constant of motion anymore, since it results to be: $A(s) = \epsilon e^{-\gamma s} = \bar{\epsilon}(s)$.

The above remarks allow us to immediately generalize our results to the case in which both *deterministic* (RF potential well plus radiation damping) and arbitrary *stochastic* effects are taken into account. In this case, in fact, the system dynamics is assumed to be ruled by

$$i\eta\tilde{\epsilon}(s) \frac{\partial}{\partial s} \Psi(x, s) = - \frac{\eta^2 \tilde{\epsilon}^2(s)}{2} \frac{\partial^2}{\partial x^2} \Psi(x, s) + \frac{1}{2} K x^2 \Psi(x, s) \quad , \quad (19)$$

where now $\tilde{\epsilon}(s)$ is an arbitrary function of s , to be specified in correspondence of the particular stochastic effects considered, but satisfying the initial condition $\tilde{\epsilon}(0) = \epsilon$. Also in this more general case of damped harmonic oscillator, we are able to give a complete set of normalized solutions of Eq. (19)

$$\Psi_m(x, s) = \frac{1}{\{2\pi 2^{2m} (m!)^2 \sigma^2(s)\}^{1/4}} \exp \left\{ -\frac{x^2}{4\sigma^2(s)} \right\} H_m \left(\frac{x}{\sqrt{2}\sigma(s)} \right) \times \exp \left\{ i \frac{x^2}{2\eta \tilde{\epsilon}(s) \rho(s)} + i(1+2m)\phi(s) \right\} , \quad (20)$$

where now the function $\sigma(s)$ satisfies the following equation

$$\frac{d^2 \sigma(s)}{ds^2} + \Gamma(s) \frac{d \sigma(s)}{ds} + K \sigma(s) - \frac{\eta^2 \tilde{\epsilon}^2(s)}{4 \sigma^3(s)} = 0 , \quad (21)$$

with $\Gamma(s) \equiv -d \log[\tilde{\epsilon}(s)]/ds$, and $\rho(s)$ and $\phi(s)$ have the same definition of (11) and (12) in terms of $\sigma(s)$. By defining $\tilde{\sigma}(s)$ as

$$\tilde{\sigma}(s) \equiv \sigma(s) \exp \left\{ \frac{1}{2} \int_0^s \Gamma(s') ds' \right\} = \sigma(s) \sqrt{\frac{\epsilon}{\tilde{\epsilon}(s)}} , \quad (22)$$

Eq. (21) becomes

$$\frac{d^2 \tilde{\sigma}(s)}{ds^2} + \overline{K}(s) \tilde{\sigma}(s) - \frac{\eta^2 \epsilon^2}{4 \tilde{\sigma}^3(s)} = 0 , \quad (23)$$

where $\overline{K}(s) \equiv K - \frac{\Gamma^2(s)}{4} - \frac{1}{2} \frac{d\Gamma(s)}{ds}$.

Some physical consideration are in order

1) At the early time ($\gamma s \ll 1$), $\tilde{\epsilon} \simeq \epsilon$, and $\sigma(s) \simeq \tilde{\sigma}(s)$. We physically expect that, during this time scale, the damping rate $\Gamma(s)$ is maximum whereas the quantum excitation is negligible. In fact, at the beginning, due to the small number of produced photons, the photon noise is negligible. Consequently, $\Gamma(s) \simeq \gamma = \text{const.}$, and, thus, $\overline{K}(s) \simeq K - (\gamma^2/4) = \text{const.}$ In particular, for $K > (\gamma^2/4)$, \overline{K} is a positive constant, so that $\tilde{\sigma}(s)$ and $\sigma(s)$ are limited functions.

2) For very large time ($\gamma s \gg 1$), since in the usual description of the circular accelerating machines [2] a sort of asymptotic equilibrium is reachable, we can seek this equilibrium by assuming that $\Gamma(s)$ vanishes as s increases its values. This physically means that, as s grows, the radiation emission produces more and more photons which, in turn, increase the quantum excitation. Since the radiation damping provides a decreasing of the r.m.s. emittance, a sort of competition between this damping and quantum excitation is thus established in such a way to reach an asymptotic limit for $\tilde{\epsilon}(s)$, say ϵ_D . Consequently, from the asymptotic condition $\Gamma(\infty) = 0$, it follows that $\overline{K} = K > 0$. Thus, we conclude that, for $\gamma s \gg 1$, $\tilde{\sigma}(s)$ is required to be limited, and from (22) and (23) it follows that asymptotic equilibrium solutions $\tilde{\sigma}(\infty)$ and $\sigma(\infty)$ for $\tilde{\sigma}(s)$ and $\sigma(s)$, respectively, exist and results

$$\tilde{\sigma}(\infty) = \sigma(\infty) \sqrt{\frac{\epsilon}{\epsilon_D}} . \quad (24)$$

Since the present description holds for an arbitrary form of $\tilde{\epsilon}(s)$, on the basis of the above physical considerations we have to assume its explicit behaviour or, equivalently,

to assume the form of $\Gamma(s)$, in such a way to recover the results of conventional theory [2]. To this end, we observe that (18), which is valid in the case of negligible quantum excitation ($\epsilon_D = 0$), suggests to assume in the more general case $\epsilon_D \neq 0$, the following evolution equation for $\tilde{\epsilon}(s)$

$$\frac{d\tilde{\epsilon}}{ds} + \gamma\tilde{\epsilon} = \gamma\epsilon_D \quad , \quad (25)$$

with $\tilde{\epsilon}(0) = \epsilon$. Consequently, we have

$$\tilde{\epsilon}(s) = \epsilon e^{-\gamma s} + \epsilon_D(1 - e^{-\gamma s}) \quad , \quad (26)$$

and

$$\Gamma(s) = \frac{(\epsilon - \epsilon_D)\gamma}{(\epsilon - \epsilon_D) + \epsilon_D e^{\gamma s}} \quad . \quad (27)$$

We observe that for $\gamma s \gg 1$, (19) becomes

$$i\eta\epsilon_D \frac{\partial}{\partial s} \Psi(x, s) = -\frac{\eta^2 \epsilon_D^2}{2} \frac{\partial^2}{\partial x^2} \Psi(x, s) + \frac{1}{2} K x^2 \Psi(x, s) \quad , \quad (28)$$

and the uncertainty relation at the minimum gives

$$\sigma(\infty)\sigma_p(\infty) = \frac{\epsilon_D}{2} \quad , \quad (29)$$

where $\sigma_p(\infty) \equiv \lim_{s \rightarrow \infty} \langle \hat{p}^2 \rangle$. Note that $\sigma_p(\infty)$ represents the expectation value of the longitudinal momentum spread of the bunch at the asymptotic equilibrium. This means that its explicit determination does not depend on the history which led the system to the equilibrium condition, but it depends on the quantum fluctuations (photon noise) only. Furthermore, $\sigma(\infty)$ can be explicitly determined by imposing the asymptotic equilibrium in the (21). Thus

$$\sigma^2(\infty) = \frac{|\eta|\epsilon_D}{2\sqrt{K}} \quad , \quad (30)$$

which, by using (29), becomes

$$\sigma(\infty) = \frac{|\eta|R_0}{\nu_s} \sigma_p(\infty) \quad , \quad (31)$$

where ν_s (synchrotron number) [1] stands for the ratio between $\Omega_s \equiv c\sqrt{K}$ [1] (synchrotron frequency) and the revolution frequency $\omega_0 = \beta c/R_0 \approx c/R_0$. Consequently, since from (4) the equilibrium value $(\overline{R^2})_{eq}$ of $\overline{R^2}$ is Q_p/γ , and since it is proved that the equilibrium energy spread is given by $\frac{1}{2}(\overline{R^2})_{eq}$ [2], it is very easy to recognize that

$$\sigma_p^2(\infty) = \frac{(\overline{R^2})_{eq}}{2E_0^2} = \frac{55}{48\sqrt{3}} \frac{\hbar c r_e \gamma_0^6}{\gamma E_0 R_0 \rho_0^2} \quad . \quad (32)$$

Hence, by combining (32) and (31) we finally obtain

$$\sigma(\infty) = \frac{|\eta|R_0}{\nu_s} \left[\frac{(\overline{R^2})_{eq}}{2E_0^2} \right]^{1/2} \quad . \quad (33)$$

Eq. (31) recovers the well-known proportionality between the equilibrium bunch length $\sigma(\infty)$ and the corresponding equilibrium momentum spread $\sigma_p(\infty)$, given in literature,

whilst (32) and (33) give explicitly these values in terms of the quantum fluctuations. We remark that, as $\sigma(s)$ and $\sigma_p(s)$ go to the asymptotic equilibrium values, $\tilde{\epsilon}$ goes to the minimum value ϵ_D of the r.m.s. emittance starting from the initial value ϵ , according to the scaling law (26). We point out that the reduction of $\tilde{\epsilon}(s)$ is due to the bunch cooling produced by the radiation damping, and, consequently, ϵ_D represents the limit value of $\tilde{\epsilon}(s)$ for which the bunch thermalization is completed. In addition, by combining (29), (32) and (33) we immediately get the expression of ϵ_D in terms of electron Compton's wavelength $\lambda_e \equiv \hbar/m_e c$ (m_e being the electron rest mass)

$$\epsilon_D = \left[\frac{55}{24\sqrt{3}} \frac{|\eta| r_e \gamma_0^5}{\nu_s \gamma \rho_0^2} \right] \lambda_e \equiv \chi_D \lambda_e . \quad (34)$$

In table 1 we report the values of the factor χ_D for some electron-positron circular machines. Note that the large values found for χ_D imply $\epsilon_D \gg \lambda_e$. This suggests that at the equilibrium the quantum-like behaviour of the system as a whole still corresponds to the Liouville regime (thermal equilibrium) and, according to (29), would represent a sort of macroscopical coherence. On the other hand, the quantum limit of such a coherence is, of course, recovered for $\epsilon_D \approx \lambda_e$ (Heisenberg regime).

In this letter we have presented an extension of the recently proposed *thermal wave model for particle dynamics* [3] to the longitudinal motion in circular accelerating machines when both RD and QE are taken into account. In this framework, the particle dynamics in the presence of a RF potential well is governed by a 1-D Schrödinger-like equation for a complex wave function, whose squared modulus gives the longitudinal bunch profile. We have shown that the solutions for the BWF of this problem are given in terms of the well-known Gauss-Hermite modes. In particular, the fundamental mode (lowest-energy mode) gives a pure Gaussian space-distribution for the particles, and the corresponding envelope equation gives an asymptotic value for the bunch length, which is expressed in terms of the quantum fluctuations (noise). Correspondingly, according to (26), as the beam is cooling, due to RD, the emittance goes to the equilibrium value ϵ_D which represents the equilibrium limit and plays the role of thermalization value.

In conclusion, the above results allow us to apply the thermal wave model, already successfully applied to the undamped longitudinal dynamics (protons) [8], to the synchrotron electron motion in a more accurate way, since in this case both radiation damping and quantum excitation are not negligible. This occurs, for example, for electrons in circular accelerating machines.

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Table 1.

machines	ρ_0 (m)	γ_0	$ \eta \approx \alpha_c$	ν_s	γ (m^{-1})	χ_D
DAΦNE	1.4	$1.02 \cdot 10^3$	$0.52 \cdot 10^{-2}$	$7.63 \cdot 10^{-3}$	$3.74 \cdot 10^{-7}$	$3.8 \cdot 10^6$
EPA	1.43	$1.2 \cdot 10^3$	$3.3 \cdot 10^{-2}$	$1.55 \cdot 10^{-3}$	$1.0 \cdot 10^{-7}$	$9.7 \cdot 10^8$
SLAC B-FACTORY	165	$1.8 \cdot 10^4$	$2.44 \cdot 10^{-3}$	$5.22 \cdot 10^{-2}$	$3.6 \cdot 10^{-7}$	$3.4 \cdot 10^7$
LEP	$3.1 \cdot 10^3$	$1.1 \cdot 10^5$	$2.9 \cdot 10^{-4}$	$8.9 \cdot 10^{-2}$	$3.54 \cdot 10^{-7}$	$5.7 \cdot 10^7$